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On a model representation for certain spatial-resonance phenomena

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It has been observed that standing surface waves in water may be excited by acoustic fields of very much higher frequency. No special relationship between the two frequencies appears to be required, but there is such a relationship between the spatial variations of the acoustic and surface wave modes. Another requirement is that the lower frequency should be comparable with the resonant bandwidth of the acoustic response of the system. An explanation of such phenomena is proposed and is tested on a somewhat idealized model by the use of techniques which could be extended to deal with more realistic situations. The model serves to explain qualitatively the available experimental observations. It is suggested that the phenomenon of spatial resonance is not confined to the interaction between water waves and acoustic fields, but may occur generally in systems having modes with related spatial patterns but greatly different frequencies.

1. Introduction

The authors have become aware of two examples where acoustic fields serve to generate standing surface waves of large amplitude in water. These waves have frequencies lower by about two orders of magnitude than that of the acoustic field driving them. In some experiments conducted at Oxford University by Dr R.E. Franklin and described to the authors by Dr J. Ockenden a glass cylinder was mounted with its axis horizontal and partially filled with water. Under certain circumstances, when the air in the cylinder was excited in a natural organ-pipe mode, standing waves of small wave-number were excited in the water. An apparently similar phenomenon was observed by Huntley (1972) in the course of experiments which involved the excitation of a beaker of water in its natural bell modes. Here too, large amplitude waves of very much lower frequency were observed to be generated. Waves with various mode patterns could occur and some of these could be maintained indefinitely. Moreover, different surface wave patterns could be generated by exciting higher order bell modes.

It might be considered that these were examples of parametric resonance but this explanation appears to be ruled out for two reasons. First, the phenomena are not sensitive to the frequency ratio of the two natural modes. In the Oxford experiment no care has to be taken in the selection of the depth of water, which

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affects the frequency of the water waves. Likewise, in Huntley's experiments the properties of the beaker, and hence the frequency of the bell mode, did not appear to be in any way critical. Second, parametric resonance over such a large frequency ratio, and for relatively small driving amplitudes, would involve extremely slow growth rates in the absence of dissipation. For the systems under consideration it is inconceivable that these growth rates could overcome the natural damping of the surface waves. Thus parametric resonance does not appear to be a likely explanation. It seems that both these phenomena must be explained in terms of resonant nonlinear coupling, of a form that so far does not appear to have been described in the literature.

The following mechanism is suggested as the explanation of such phenomena and its investigation is the object of this paper. Because the acoustic field is driven at an angular frequency ω close to resonance for a particular mode, whose spatial variation over the water surface is described by $X(x)$, this mode may be expected to dominate the acoustic field. Further, let there be present a small amount of a standing surface wave whose spatial variation over the surface is given by $Y(x)\,e^{i\sigma t}$. For the mechanism under consideration it is necessary that $[X(x)]^2$, when expanded in a modal decomposition, has a significant amount of the mode $Y(x)$ and that $X(x)\,Y(x)$ has a significant amount of the mode $X(x)$. These are certainly justified assumptions for Huntley's experiments but whether this is so for the Oxford configuration is not known to us. The acoustic field drives a surface wave field of the form $X(x)\,e^{i\omega t}$, but this is of quite small amplitude because its frequency is so much greater than the natural frequency of standing waves with the modal form $X(x)$. The nonlinear surface conditions couple the two standing waves, so that very small amounts of the modes $X(x)\,e^{i(\omega\pm\sigma)t}$ appear in the surface displacement. This is a direct consequence of the assumption about the modal decomposition of XY. Now if at least one value of $\omega\pm\sigma$ lies within the resonant bandwidth of the acoustic mode $X(x)$ the acoustic field will contain the modal response $X(x)\,e^{i(\omega\pm\sigma)t}$. Moreover, the resonant amplification of the surface displacement at one or both of these frequencies may compensate for the very small surface response of the type $X(x)\,e^{i(\omega\pm\sigma)t}$. Nonlinear coupling between the acoustic modes $X(x)\,e^{i\omega t}$ and $X(x)\,e^{i(\omega\pm\sigma)t}$, and the assumption about the modal decomposition of X^2, imply a very small forcing of the form $Y(x)\,e^{i\sigma t}$ on the water surface. However, this is a natural mode of the surface wave system, so there is a greatly enhanced response to such forcing. This may be seen to complete a cycle, in that the presence of some surface wave mode may lead to the generation of the same mode through interaction with the acoustic field. Thus there is a potentiality for instability, but to confirm the instability it is necessary to show that the phase relationships are appropriate for a significant energy feed into the standing surface waves. There are certain features of Huntley's observations which suggest that the above explanation is correct. First, all the surface wave modes he has observed for a variety of modal shapes excited in the acoustic field are consistent with the assumed properties of X^2 and XY. Huntley has also observed that when the surface waves are established the acoustic response may beat with angular frequency σ or 2σ. This may be explained by assuming the presence of one or both of the angular frequencies $\omega\pm\sigma$.

Under these circumstances a detailed investigation of this possible spatial-resonance interaction is justified, but the complexity of any calculation is rather daunting. It may be noted that as many as four modes may contribute to the significant part of the calculations and that each of these must be considered in both the acoustic and surface wave fields. Moreover, the mechanism envisaged involves the enhancement of certain very small terms by large resonant amplification. This happens in both the acoustic and surface wave fields and hence great care must be taken to obtain certain critical terms to a high order of accuracy. Huntley's configuration seems to offer the better opportunities for accurate experimental investigation but it is rather unattractive for a preliminary theoretical investigation of the proposed mechanism. As a first stage it seemed appropriate to investigate a model with simple eigenfunctions, which does not involve the coupling between the water and the elastic vibrations of the containing vessel. Nevertheless, it was intended that, if the pattern of calculation proved successful the analysis would eventually be extended so that a direct comparison could be made between theory and experiment. Thus although the model chosen is rather close to the Oxford experiments, the calculations have been developed with an eye on Huntley's configuration.

The configuration chosen for theoretical investigation is a rectangular organ pipe filled with water of sufficient depth for it to be effectively infinite for standing surface waves of the wavelength under consideration. Moreover, it will be assumed that the flow field is two-dimensional. Even this simplified model problem has a host of small parameters, some of which are sufficiently different in size to make it extremely difficult to work *ab initio* on any soundly based hierarchy of scales. This rapidly becomes apparent once calculations start, because a variety of products and quotients of small parameters arise. Thus some rationale has to be settled for dealing with small parameters such as the dimensionless numbers measuring the importance of nonlinear effects in the acoustic and surface wave fields: the small dissipation parameters (different in order of magnitude for the two fields), the ratio of the frequency of surface waves to that of the driving acoustic field, the ratio of the density of air to that of water, and the relative departure of the driving frequency from acoustic resonance conditions. Our aim has been to retain the largest term with any specified physical effect. In making this assessment we have been guided by the results for Huntley's experiments.

In particular, it will be assumed that the acoustic response at resonance is limited by dissipation rather than nonlinear detuning. It is highly probable that in Huntley's experiments the displacements of the container are small enough to be considered as infinitesimal. However, the situation with regard to the Oxford experiments is not so clear. Chester (1964) has shown that nonlinear effects can be extremely important in organ-pipe modes, fundamentally because sound waves are not dispersive. However, as is shown in the later analysis, the presence of the water surface introduces a small dispersive element into the acoustic field so that the relevance of Chester's shock formation mechanism is not clear. It might be added that if the acoustic resonance is in fact limited by nonlinear, rather than dissipative, effects the present theory could be modified but the analysis would be much more complex (see appendix).

In the model calculation the dissipations will be introduced through empirical factors. In a confined region dissipation in the acoustic field will be largely due to losses in the boundary layers; such losses usually produce attenuation rates and frequency shifts of the same small order. To include both effects in the analysis would add tremendously to the difficulty of the calculations. The following scheme is proposed to avoid this in a way which permits an empirical adjustment for comparison with experiment. The acoustic field will be assumed to satisfy the wave equation modified by the addition of a linear term in the first time derivative. The small coefficient multiplying this term will be chosen so that the modified equation predicts the experimentally observed decay rate for the mode which is acoustically resonant. Such an equation will not describe the decay of other modes correctly, but this is unimportant for in the calculations of the acoustic field it is only in the equation for the resonant mode that any such small terms play a significant role. The frequency shift will be included in the eventual analysis by making the observed frequency of maximum response correct. Such a correction needs to be made only if one is concerned with the exact position on the frequency scale of the resonance phenomena. Similarly, an artificial damping will be introduced into the surface wave calculations so as to give the correct decay rate for the observed standing wave. Because the calculations are not sensitive to the frequency of this standing wave, no effort has been made to include any small frequency shift.

In any actual problem in which theory is to be compared with experiment it would be possible to introduce a soundly based system of approximations in which the assessment of relative sizes could be made rationally. It may be that certain terms which we have neglected are important in some circumstances, but every endeavour has been made to retain significant terms. From the pattern established by the present calculation, it will be much easier to make an assessment of how significant different types of terms will be in any calculation concerning an experimental configuration. It is an interesting feature of the calculations that the properties of the model system under investigation do not play a significant role in the explanation of the instability mechanism, or in the determination of the ultimate state which develops. From the form of the mathematics it can be seen that similar phenomena might be expected in many weakly nonlinear systems with geometrically related modes but greatly differing frequencies.

2. Acoustic field

Here we consider the behaviour of an acoustic field driven at a nearly resonant frequency at one boundary, while another of its boundaries moves in a certain prescribed way which will later be identified with the motion of the water surface. Consider a horizontal column of air of density ρ_a, length l and depth d, driven at one vertical end by a periodic motion of angular frequency ω. Let the x axis be taken along the column and the z axis vertically upwards with its origin in the undisturbed water surface. Let the disturbed water surface be denoted by $\zeta(x, t)$, the velocity potential by $\Phi(x, z, t)$ and the pressure by $p(x, z, t)$, where t is time.

Then, as explained earlier, the acoustic field will be assumed to be describable by

$$c^2\nabla^2\Phi = \Phi_{tt} + 2\nu\Phi_t, \tag{1}$$

where c is the sound speed in the undisturbed air and ν is the observed (logarithmic) decay rate of free acoustic waves in the resonant mode. The boundary conditions will be imposed in the form

$$\Phi_z + \Phi_{zz}\zeta = \zeta_t + \Phi_x\zeta_x \quad \text{on} \quad z = 0, \tag{2}$$

which is the kinematic surface condition correct to second order in the surface displacement, and

$$\Phi_z = 0 \quad \text{on} \quad z = d,$$

$$\Phi_x = 0 \quad \text{on} \quad x = l,$$

$$\Phi_x = \{A\, e^{i\omega t} + *\} \quad \text{on} \quad x = 0,$$

where A is the (very small) driving amplitude and $*$ denotes the complex conjugate of all terms written explicitly within the bracket. Some objection may be raised to combining the linearized equation (1) with the second-order boundary condition (2). The reason for this form of approximation is as follows. The only second-order terms which are significant in this part of the calculation are the acoustically resonant terms, which have a large response to small drive. The biggest such terms come from products of the large acoustically resonant mode with the large surface displacement in the surface wave mode, i.e. the terms retained in (2). The nonlinear terms in the field equation would yield terms of the same form but significantly smaller, since the acoustic response in the surface wave mode is very much reduced because of the small frequency ratio.

It will be assumed that the driving frequency is close to the resonance frequency of the natural mode, which is z-independent and varies with x as $\cos(m\pi x/l)$. Further, it is assumed that no other acoustic mode is nearly resonant at this frequency, so that the acoustic field will be predominantly in the resonant mode. The square of this spatial dependence is thus dominated by x-independent modes and a $\cos(2mx/l)$ dependence on the surface. By our original argument about $X(x)$ and $Y(x)$, surface waves generated must involve $\cos(2mx/l)$, since an x-independent heaving of the water surface is incompatible with conservation of mass. Thus, for the possible mechanism described in the introduction, it suffices to consider the acoustic field in the form

$$\Phi = \cos(m\pi x/l)\{\Phi_0\, e^{i\omega t} + \Phi_+\, e^{i(\omega+\sigma)t} + \Phi_-\, e^{i(\omega-\sigma)t} + *\} \\ + \cos(2m\pi x/l)\{\Psi\, e^{i\sigma t} + *\} + \Phi', \tag{3}$$

where Φ' includes all other modes which are unrelated to a mode which is near resonant in either the acoustic or surface wave field. The various Φ's and Ψ must be considered as complex to allow for as yet undetermined phase differences. Further, they must be regarded as slowly varying functions of time, since we are interested in energy transfers between the various modes. No assumption is made about the relative sizes of the driven mode Φ_0 and the system generated modes Φ_+, Φ_- and Ψ since we wish to compute the final configuration as well as explain the instability. It is perhaps worth repeating that the explicit Φ's may be expected to be large in comparison with Ψ because they are the resonant modes in the acoustic system while the Ψ, driven by a relatively large surface displacement, has too small a frequency to be significantly generated in the acoustic field. Throughout the analysis it will be assumed that Φ', which is nowhere resonant and hence small, will not contribute significantly to any nonlinear interaction.† Moreover, its linear response is of little interest so that such terms will be ignored in the subsequent calculation. A similar argument suggests writing the surface displacement of the water in the form

$$\zeta = \cos(m\pi x/l)\{\zeta_0 e^{i\omega t} + \zeta_+ e^{i(\omega+\sigma)t} + \zeta_- e^{i(\omega-\sigma)t} + *\} + \cos(2m\pi x/l)\{\eta e^{i\sigma t} + *\} + \zeta'. \tag{4}$$

In this case it is the η term which is expected to be relatively large, since the other terms are driven by acoustic pressure fields of frequencies far removed from the gravity wave resonance.

The calculational procedure envisaged is the substitution of a Fourier decomposition

$$\Phi = \sum_{k,n=0}^{\infty} \cos(k\pi x/l)\cos(n\pi z/d)\,\Phi_{kn}(t)$$

into the damped wave equation (1). Because the boundary conditions on $x = 0$ and $z = 0$ are not consistent with this form of expansion, $\nabla^2\Phi$ is not given correctly by formally differentiating the Fourier decomposition. This may be corrected by a method described in Jeffreys & Jeffreys (1962, p. 441). Alternatively, use may be made of finite Fourier transforms. By either method it may be shown that

$$\nabla^2\Phi = \sum_{k,n=0}^{\infty} -\{(k\pi/l)^2 + (n\pi/d)^2\}\cos(k\pi x/l)\cos(n\pi z/d)\,\Phi_{kn} - 2l^{-1}[\Phi_x]_{x=0} - d^{-1}[\Phi_z]_{z=0} + R', \tag{5}$$

where R' denotes terms which are identically zero for the modes $n = 0$ and $k = m$ or $2m$, which are the only modes for which detailed calculations are to be made. Equation (1) for the damped acoustic field may now be applied to yield equations for the temporal variations of Φ and Ψ. Consider first the equation for Ψ. Since Ψ is much smaller than the other terms in the acoustic field, the even smaller terms such as Ψ_t and $\nu\Psi$ will be neglected. This is the justification for using the one damping coefficient for all frequencies, because it is only in one narrow frequency band that the damping contributes significantly to the calculations. Thus one obtains as a reasonable approximation

$$\Psi = -i\sigma\eta/d(2m\pi/l)^2 \tag{6}$$

for the relative small acoustic response in this mode.

The calculations for the three nearly resonant terms have to be made rather more carefully, because of the amplified response to small driving terms. For such terms we make the approximation

$$\frac{\partial^2}{\partial t^2}\{\Phi_0 e^{i\omega t}\} + 2\nu\frac{\partial}{\partial t}\{\Phi_0 e^{i\omega t}\} \simeq \{-\omega^2\Phi_0 + 2i\omega\Phi_{0t} + 2i\nu\omega\Phi_0\}e^{i\omega t}, \tag{7}$$

† In Chester's (1964) theory of the organ pipe the higher harmonics are also natural modes of the system, since the wave equation is non-dispersive, and this assumption does not apply (see appendix).

where terms such as $\nu\Phi_{0t}$ and Φ_{0tt} have been dropped in comparison with $i\omega\Phi_{0t}$. It will be seen later that small phase changes are unlikely to alter the essential features of the predictions, so that these seem reasonable approximations. Then, combining (1), (3), (4), (5) and (7) leads to the equation

$$\Phi_{0t}+\nu\Phi_0+\tfrac{1}{2}i\omega^{-1}\{\omega^2-(m\pi c/l)^2\}\,\Phi_0+\tfrac{1}{2}c^2d^{-1}\zeta_0$$
$$= ic^2l^{-1}\omega^{-1}A+\tfrac{1}{2}i(cm\pi/l)^2\,(d\omega)^{-1}\,(\Phi_+\eta^*+\Phi_-\eta) \qquad (8a)$$

for the driven mode. In this arrangement every term is small because the fact that ω is close to the resonant frequency of the mode makes the last term on the left-hand side small. Almost identical equations are similarly obtained for the side-band near-resonant modes. These are

$$\Phi_{+t}+\nu\Phi_++\tfrac{1}{2}i\omega^{-1}\{(\omega+\sigma)^2-(m\pi c/l)^2\}\,\Phi_++\tfrac{1}{2}c^2d^{-1}\zeta_+ = \tfrac{1}{2}i(cm\pi/l)^2\,(d\omega)^{-1}\,\Phi_0\eta, \qquad (8b)$$
$$\Phi_{-t}+\nu\Phi_-+\tfrac{1}{2}i\omega^{-1}\{(\omega-\sigma)^2-(m\pi c/l)^2\}\,\Phi_-+\tfrac{1}{2}c^2d^{-1}\zeta_- = \tfrac{1}{2}i(cm\pi/l)^2\,(d\omega)^{-1}\,\Phi_0\eta^*, \qquad (8c)$$

where in all but the one large term the differences between ω and $\omega+\sigma$ have been neglected. It is for the same reason that the slight variation of the damping with frequency within the resonant bandwidth has been neglected in forming equation (1).

3. Surface-wave field

In this section the response of the water surface to the acoustic pressure will be calculated. For the irrotational homentropic motion of the air, Bernoulli's equation gives

$$\int\frac{dp}{\rho}+\Phi_t+V+\tfrac{1}{2}\{\Phi_x^2+\Phi_z^2\} = \text{constant},$$

where V denotes the potential of the conservative external force field. It can be shown, therefore, that the pressure p_s on the surface is given, correct to second order in small quantities, by

$$p_s = -\rho_a\{\Phi_t+g\zeta+\Phi_{tz}\zeta+\tfrac{1}{2}(\Phi_x^2+\Phi_z^2)-\tfrac{1}{2}(\gamma-1)/(\gamma c)^2\,(\Phi_t+g\zeta)^2\},$$

where all derivatives of Φ are to be evaluated on $z = 0$, ρ_a denotes the density of air and γ the adiabatic index. The expansions (3) and (4) are to be substituted into this equation, and once again only certain second-order terms are retained. This expression for the pressure is to be used to calculate the response of the water, so that now the only second-order terms retained are those of angular frequency σ and spatial variation $\cos(2m\pi x/l)$ on the pressure. Again only the biggest terms of a given form are retained, so that, to sufficient accuracy, the surface pressure may be represented by

$$p_s = -\rho_a\cos(m\pi x/l)\,[i\omega\Phi_0e^{i\omega t}+i\omega\Phi_+e^{i(\omega+\sigma)t}+i\omega\Phi_-e^{i(\omega-\sigma)t}+*]-\rho_a\cos(2m\pi x/l)$$
$$\times\{e^{i\sigma t}[i\sigma\Psi+g\eta-\tfrac{1}{2}((m\pi/l)^2-(\gamma-1)\,\omega^2/(\gamma c^2))(\Phi_0\Phi_-^*+\Phi_0^*\Phi_+)]+*\}. \qquad (9)$$

Note that terms like $\Phi_0\zeta_-^*$ are indeed small in comparison with the nonlinear terms retained, as would be contributions from second-order terms in the acoustic field equations.

Let ϕ denote the velocity potential of the irrotational flow in the water. Then ϕ may be represented in the form

$$\phi = e^{m\pi z/l}\cos(m\pi x/l)\{\phi_0 e^{i\omega t}+\phi_+ e^{i(\omega+\sigma)t}+\phi_- e^{i(\omega-\sigma)t}+*\} + e^{2m\pi z/l}\cos(2m\pi x/l)\{\psi e^{i\sigma t}+*\}+\phi',$$

where the notation is similar to that employed for the acoustic field. The surface boundary conditions to be satisfied on $z=0$ by this velocity potential are, correct to second order,

$$\phi_z+\phi_{zz}\zeta = \zeta_t+\phi_x\zeta_x,$$

$$\rho\{\phi_t+g\zeta+\tfrac{1}{2}(\phi_x^2+\phi_z^2)+\phi_{tz}\zeta\}+p_s = 0,$$

where the surface pressure is given by (9). For the modes which are acoustically resonant, and hence little excited in the surface waves, the linear approximations alone suffice. Thus, to an approximation consistent with the procedure we have followed, it is easy to show that

$$\phi_{0,\pm} = (\rho_0/\rho)\,\Phi_{0,\pm} \tag{10a}$$

and

$$\zeta_{0,\pm} = -i\omega^{-1}(m\pi/l)\,(\rho_a/\rho)\,\Phi_{0,\pm}, \tag{10b}$$

where this notation implies that the equations apply to all three frequencies.

For the resonant standing waves in the water, a better approximation retaining second-order terms is used, but once again only the largest second-order terms of a given form are kept. As far as the resonant surface wave is concerned, the kinematic surface condition is sufficiently well approximated by

$$(2m\pi/l)\,\psi = i\sigma\eta+\eta_t, \tag{11a}$$

since equations (10) imply that the second-order terms neglected here are estimated in size by $(\rho_a/\rho)^2\Phi_0\Phi_-^*$, typically, and these are small in comparison with other terms arising in the complete treatment of the surface boundary condition. The dynamic boundary condition on the surface yields

$$(i\sigma\psi+\psi_t)+g\eta = (\rho_a/\rho)\{i\sigma\Psi+g\eta-\tfrac{1}{2}[(m\pi/l)^2-\omega^2(\gamma-1)/\gamma c^2]\,(\Phi_0\Phi_-^*+\Phi_0^*\Phi_+)\}, \tag{11b}$$

and ψ may be eliminated from (11). Thus one obtains

$$\eta_t+\tfrac{1}{2}i\sigma^{-1}\{\sigma^2-2m\pi g/l\}\,(1-\rho_a/\rho)\,\eta-(\rho_a/\rho)\,(m\pi/l)\,\Psi = \tfrac{1}{2}i\sigma^{-1}(\rho_a/\rho)\,(m\pi/l)\,\{(m\pi/l)^2-\omega^2(\gamma-1)/\gamma c^2\}\,(\Phi_0\Phi_-^*+\Phi_0^*\Phi_-). \tag{12}$$

Equations (6), (8), (10) and (12) may now be combined into a set of ordinary differential equations:

$$\Phi_{0t}+\nu\Phi_0+\tfrac{1}{2}i\omega^{-1}\{\omega^2-(m\pi c/l)^2-(\rho_a/\rho)\,c^2d^{-1}(m\pi/l)\}\,\Phi_0 = ic^2l^{-1}\omega^{-1}A+\tfrac{1}{2}i(cm\pi/l)^2\,d^{-1}\omega^{-1}(\Phi_+\eta^*+\Phi_-\eta), \tag{13a}$$

$$\Phi_{+t}+\nu\Phi_++\tfrac{1}{2}i\omega^{-1}\{(\omega+\sigma)^2-(m\pi c/l)^2-(\rho_a/\rho)\,c^2d^{-1}(m\pi/l)\}\,\Phi_+ = \tfrac{1}{2}i(cm\pi/l)^2\,d^{-1}\omega^{-1}\Phi_0\eta, \tag{13b}$$

$$\Phi_{-t}+\nu\Phi_{-}+\tfrac{1}{2}i\omega^{-1}\{(\omega-\sigma)^2-(m\pi c/l)^2-(\rho_a/\rho)\,c^2 d^{-1}(m\pi/l)\}\,\Phi_{-}$$
$$=\tfrac{1}{2}i(cm\pi/l)^2\,d^{-1}\omega^{-1}\Phi_0\eta^*, \qquad (13\,c)$$

$$\eta_t+\nu'\eta+\tfrac{1}{2}i\sigma^{-1}\{\sigma^2-(2m\pi g/l)\,(1-\rho_a/\rho)+(\rho_a/\rho)\,g\,d^{-1}\}\,\eta+i\sigma(m\pi/l)^2\eta^*\eta^2$$
$$=\tfrac{1}{2}i\sigma^{-1}(\rho_a/\rho)\,(m\pi/l)\,\{(m\pi/l)^2-\omega^2(\gamma-1)/\gamma c^2\}\,(\Phi_0\,\Phi_-^*+\Phi_0^*\,\Phi_+), \qquad (13\,d)$$

where two additional terms have been included in the last equation. The first of these, $\nu'\eta$, is to represent the damping of the surface waves, and it is intended that the value of ν' should be obtained from the experimentally determined decay rate of a pure standing wave of this modal shape. The second term added is the last term on the left-hand side and has been included because experiments suggest that the surface waves generated are far from small in amplitude. The actual value of the coefficient is taken from the calculations for a standing wave by Tadjbakhsh & Keller (1960). The inclusion of such a term will not affect the explanation of the instability mechanism leading to the formation of the standing waves but it may be of significance in computing the field after the instability has developed.

It may also be observed from (13 *a*) that even if the surface wave does not develop, i.e. $\eta = 0$, there is an additional term $(\rho_a/\rho)c^2d^{-1}(m\pi/l)$ due to the presence of the water surface. As this term is only linear in m, which is a measure of the wavenumber, the acoustic field will be slightly dispersive. As it is likely that this small dispersion is rather larger than that due to viscous and nonlinear effects, there is a smaller probability that the shock formation discussed by Chester (1964) will be important, even for strong acoustic fields.

4. Instability theory

Equations (13) have been derived for an idealized model situation, but hereafter we shall consider more general equations of a rather similar form which, it will be shown, may be expected to apply to a broader class of problems. The equations which will be considered are

$$\Phi_{0t}+\nu\Phi_0+i\Delta\Phi_0 = B+i(\alpha^*\Phi_+\eta^*+\alpha\Phi_-\eta), \qquad (14\,a)$$
$$\Phi_{+t}+\nu\Phi_+ +i(\Delta+\sigma)\Phi_+ = i\alpha\Phi_0\eta, \qquad (14\,b)$$
$$\Phi_{-t}^*+\nu\Phi_-^*+i(\sigma-\Delta)\,\Phi_-^* = -i\alpha\Phi_0^*\eta, \qquad (14\,c)$$
$$\eta_t+\nu'\eta+i(\delta+\kappa\eta\eta^*)\,\eta = i\beta(\Phi_0\,\Phi_-^*+\Phi_0^*\,\Phi_+), \qquad (14\,d)$$

where Δ is the difference between the driving frequency and the natural frequency of the acoustic field, δ is the frequency difference of the low frequency waves generated from the natural frequency of the standing waves, κ is the coefficient of nonlinear frequency shift and α and β are complex interaction coefficients. Equations (14), with α and β real and positive, reduce to equations (13), except that (13 *c*) is replaced by its complex conjugate. Moreover, (14) should apply to a much broader class of problems. Essentially what is needed is two classes of waves whose frequency ratio is large, with mode shapes satisfying the prescription discussed in the introduction and such that there is a quadratic coupling between the modes. All such systems would have equations of the same form as (14), save that the coupling coefficients in the first three equations might not be so simply related.

However, we argue that the relationships between the coefficients in the proposed equations (14) characterize a broad class of modal interactions. From recognition of the fact that the manner in which $\Phi_-\eta$ contributes to the equation for Φ_0 is identical with that in which $\Phi_0\eta$ contributes to the equation for Φ_+, it follows that the interaction coefficient in (14*b*) is the same as the second interaction coefficient in (14*a*). Similarly the interaction coefficient in (14*c*) must be the complex conjugate of the first interaction coefficient in (14*a*). These identifications are general for any system. Now the derivation of the interaction coefficients involves performing linear operations on $X(x)\,\Phi \pm e^{i(\omega\pm\sigma)t}$ and $Y(x)\,\eta\,e^{i\sigma t}$, and then picking out the coefficients of certain products. As the only operation which does not produce a real coefficient is $\partial_t e^{it}$, it follows that the contribution associated with any Φ to the interaction coefficient will be real for all problems in which the quadratic interaction terms do not contain an odd-order derivative of the high frequency field with respect to time. This is a reasonably broad class of interactions for which the interaction coefficient associated with η^* must be the complex conjugate of that associated with η. Even if this condition is not met there are additional reasons for supposing that the relation implied in (14*a*) must hold. First, if the coupling coefficients are not complex conjugates then we are faced with the remarkable situation in which spatial resonance could act in reverse, and a sufficiently strong low frequency field would drive a high frequency response. Second, equations (14) yield simple results, whereas allowing the one additional variation in the interaction coefficients leads to a lengthy catalogue of results for different relationships between the numerous coefficients.

In order to explain the onset of the instability, it suffices to note that if the system were started from a state of near rest the quadratic terms would initially be unimportant; (14*a*) then implies that the system would tend to a state in which

$$\Phi_0 = B/(\nu + i\Delta). \tag{15}$$

With this approximation for Φ_0 the remaining three equations become coupled linear constant-coefficient equations for Φ_+, Φ_- and η. Solutions with time dependence e^{ist} may be sought; this leads to the characteristic equation

$$\{(s+\sigma-i\nu)^2-\Delta^2\}\,(s+\delta-i\nu')+2\alpha\beta\Delta\Phi_0\Phi_0^* = 0. \tag{16}$$

Without the restrictive assumption made on the interaction coefficients in (14*a*) there would be an additional term involving $s\Phi_0\Phi_0^*$ and general conclusions are hard to draw. The system under investigation will be stable if all roots of the cubic equation (16) have positive imaginary parts and unstable if there is at least one root with a negative imaginary part. For small enough values of $\Phi_0\Phi_0^*$, the roots of equation (16) will be close to the values $-\delta+i\nu'$ and $+\Delta-\sigma+i\nu$, and hence no development of the surface waves is to be expected. On the other hand, for large values of $\Phi_0\Phi_0^*$, non-zero values of Δ, α and β, and moderate values of σ, ν, Δ, δ' and ν', the roots of equation (16) lie close to those of

$$s^3 = -2\alpha\beta\Delta\Phi_0\Phi_0^*.$$

Irrespective of the phases of α and β or the sign of Δ, such a system must be unstable, because at least one, and possibly two, of the roots must have a negative imaginary part. Thus the system is unstable and the low frequency mode is amplified, for strong enough forcing close enough to, but not exactly at, the high frequency resonance.

The curve of neutral stability (i.e. the graph of the relation between Δ and $|B|$ separating the regions of stability and instability) is of considerable importance. This curve has two branches whose algebraic descriptions are greatly simplified when ν' and δ are very small in comparison with both ν and σ. It will generally be the case that the high frequency field will be much more rapidly attenuated than the low frequency field. Further, the low frequency field is expected to be generated at a frequency relatively close to its resonance frequency. So such assumptions seem reasonable. Then it may be shown that, where for simplicity we have reverted to the case of equations (13) in which $\alpha\beta$ is real and positive, for $\Delta < 0$ there is one branch of the marginal stability curve given by

$$|B|^2 = -\tfrac{1}{2}\sigma(\Delta^2+\nu^2)^2/\alpha\beta\Delta, \tag{17a}$$

while for $\Delta > 0$ there is another branch given by

$$|B|^2 = \tfrac{1}{4}(\nu'/\nu)(\Delta^2+\nu^2)[(\Delta^2+\nu^2-\sigma^2)^2+4\nu^2\sigma^2]/\alpha\beta\Delta\sigma. \tag{17b}$$

The phenomenon is not to be expected unless $\omega \pm \sigma$ is within the resonance bandwidth, of which ν is a measure, so that the factor ν'/ν implies that an instability with $\Delta > 0$ can be excited for much smaller forcing of the acoustic field than an instability with $\Delta < 0$. It must be stressed that this result is a consequence of the assumption that $\alpha\beta$ is positive and is not a general property of equations (14). Sketches of the neutral-stability curves are shown in figure 1.

In Huntley's experiments the instability has only been observed when the acoustic system is driven at a frequency greater than the resonant frequency. His preliminary neutral-stability curves show a striking qualitative resemblance to the theoretical curve described by (17*b*). The quantitative agreement is also good once a suitable factor $\alpha\beta$ is chosen. With this value of $\alpha\beta$, the minimum value of $|B|$ necessary to drive the instability at frequencies below resonance conditions may be calculated from (17*a*). This has been found to be well beyond the capacity of the vibrator that Huntley has been using. However, in so far as a comparison can reasonably be made it looks as if the instability mechanism examined in this paper is exemplified by the phenomenon investigated by Huntley.

For circumstances under which instability does occur, equations (14) provide a basis for determining the style of motion which develops. Physically, it is likely that the system will either tend to some form of limit cycle or, alternatively, tend to a state in which the amplitudes in the modes involved become constant in time. In Huntley's experiments it is the second possibility which is observed to occur. If one assumes that the equations (14) admit a solution which becomes independent of time, then they provide a set of nonlinear equations for such a state. If Φ_+ and Φ_- are eliminated by the use of (14*b*) and (14*c*), and the imaginary part is taken of the transformed equation (14*d*), it may be shown that a steady

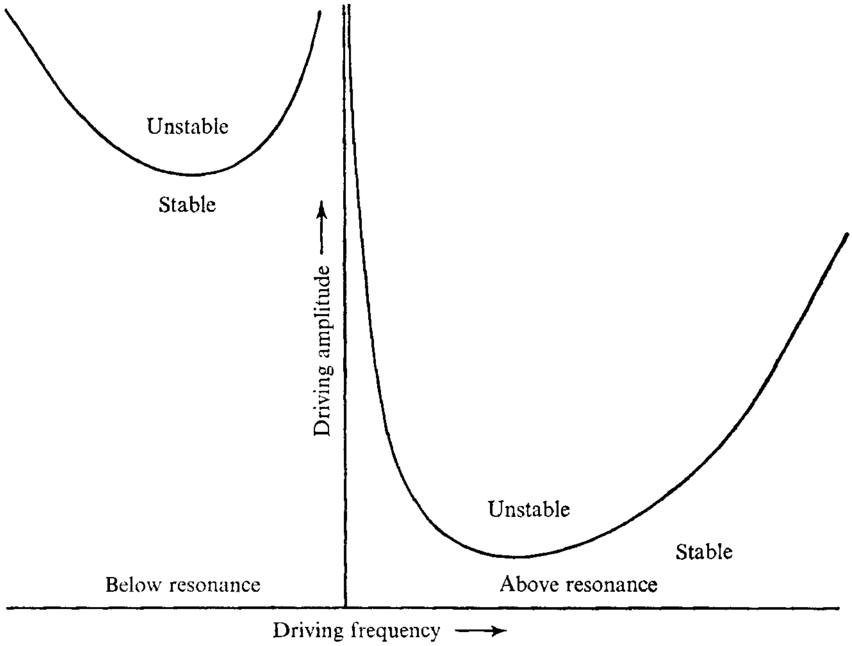

FIGURE 1. Neutral-stability curves.

amplitude response is only possible for driving frequencies above resonance. The response in the driven mode, if there is an instability in the surface wave mode, is given by

$$\Phi_0 \Phi_0^* = (\nu'/\nu)\{4\nu^2\sigma^2 + (\Delta^2+\nu^2-\sigma^2)^2\}/4\alpha\beta\sigma\Delta. \tag{18}$$

A comparison of this with (15) and (17*b*) shows that, at a given driving frequency, as the driving amplitude B is increased the response in the driven mode increases until the limit of stability is reached and thereafter remains constant. The effect of further increasing B is merely to increase the energy in the surface waves and side-band modes. The real part of the transformed equation (14*d*) yields

$$\delta + \kappa\eta\eta^* = \tfrac{1}{2}(\nu'/\nu)(\Delta^2+\nu^2-\sigma^2)\,\sigma^{-1}, \tag{19}$$

which gives a partial determination of the frequency shift of the standing surface waves. Elimination of Φ_+ and Φ_- yields the further complex equation

$$\Phi_0\{1+2\alpha^2\eta\eta^*[(\nu+i\Delta)^2+\sigma^2]^{-1}\} = B/(\nu+i\Delta). \tag{20}$$

Equations (18), (19) and (20) suffice to determine the amplitude of the surface wave, the frequency shift and the phase in the driven mode relative to the drive. One should not expect to be able to determine the phase of the surface displacement as this is related to the initial disturbance. However, apart from this, the amplitudes of the side-band modes can be calculated from the steady forms of (14*b*) and (14*c*). Thus a satisfactory theory may be considered to have been developed for the more easily excited branch of the neutral-stability curve. It is possible that even in these circumstances other limiting solutions exist, but there is no evidence from Huntley's experiments that this is so. It may be noted that many of the parameters, such as ν, ν' and the resonant frequencies, can be measured experimentally. Thus when spatial resonance is observed in circumstances where the theoretical calculations are not feasible, parameters measured from the neutral-stability curve may permit the determination of the interaction coefficients for the appropriate form of equations (14). The equilibrium theory then provides sufficient measurable quantities to permit a very stringent testing of its validity.

When one examines the structure of the calculations in §§2, 3 and 4, it is apparent that very little is dependent on the detailed physics of the situation. Equations of the form of (14) are to be expected in any closed system with weakly quadratic coupling in which there are two spatially similar waves with greatly different resonance frequencies. It is of interest to note that if the resonant bandwidth of the higher frequency mode is too narrow this form of instability cannot occur. In fact, the greater the damping in this mode the more likely it is that such an instability mechanism can occur. In view of these remarks it is interesting to speculate whether other examples of this spatial resonance may have been observed.

One possibility is reported in a paper by Ames, Lee & Zaiser (1968) on the ballooning of a travelling thread-line. Much of their work is concerned with conditions under which transverse vibrations are generated by longitudinal forcing, but the frequency ratio is not large. Under these conditions ballooning is an example of parametric resonance. However these authors commented briefly that, when the tension in the string is small (and hence the ratio of frequencies is large), a rather different form of ballooning is observed. In these circumstances the ballooning is in a mode with doubled wavenumber, and it might be that the different behaviour is associated with spatial resonance. Certainly, the coupling between longitudinal and transverse vibrations of a string is described by equations for which the general conditions for possible spatial resonance can be met for small initial strains. Another coupling which might exhibit spatial resonance is that between surface waves and the much slower internal waves in a closed region, but we know of no experimental observations. It is just possible that there are conditions under which a seiche may generate internal waves.

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Appendix

At several points in the above discussion doubts were expressed as to whether the acoustic waves in the Oxford experiments could be described accurately by a linear theory. The sound field according to linear theory, having the form $\cos(m\pi x/l)\, e^{i\omega t}$, interacts with itself to drive a mode of the form $\cos(2m\pi x/l)\, e^{i2\omega t}$; if 2ω is within the resonant bandwidth of this mode, then the sound field will contain a significant amount of the second harmonic. Likewise, the first and second harmonics interact to drive the third harmonic, and so on. For the model problem studied above, if the original sound field is driven at a frequency Δ above resonance then the nth harmonic is at a frequency

$$n\Delta + (n-1)\,(\rho_a/\rho)\,(c/d)$$

above its resonant frequency, and for typical laboratory scales this is beyond the resonant bandwidth. As the water depth is reduced, however, the sound waves become less dispersive and if the water is sufficiently shallow the nth harmonic is at a frequency only $n\Delta$ above its resonance value, so that there will be a significant amount of the higher harmonics present in the sound field. Here we outline how the calculations on the onset of instability could be modified to allow for the presence of strong harmonics in the basic sound field. A calculation of the final state would require extremely complicated mathematics, presumably akin to what was developed by Chester (1964).

We assume that the most unstable water wave mode has the spatial structure $\cos(2m\pi x/l)$ and by analogy with the preceding calculations we define Δ as the difference between the actual and natural frequencies of the $\cos(m\pi x/l)$ contribution to the basic sound field. For the onset of instability, we can regard the basic sound field as being steady, and we need to derive equations that govern the slow evolution in time of η, the amplitude of the water wave, and the amplitudes Φ_+ and Φ_- of the side-band sound waves with spatial structures $\cos(m\pi x/l)$ and frequencies respectively $\Delta+\sigma$ and $\Delta-\sigma$ above the natural frequency. The methods of §§ 2 and 3 are still applicable and the final equations are basically of the form (14*b*–*d*), with extra quadratic terms which reflect the presence of strong higher harmonics in the basic sound field. By seeking the possible quadratic forms that yield the correct spatial and temporal periodicities, we conclude that the appropriate modification of equations (14*b*–*d*) is

$$\Phi_{+t} + \nu\Phi_+ + i(\Delta+\sigma)\,\Phi_+ = i\alpha\Phi_0\eta + i\gamma\Phi_1\Phi_-^*, \tag{A 1 a}$$

$$\Phi_{-t}^* + \nu\Phi_-^* + i(\sigma-\Delta)\,\Phi_-^* = -i\alpha\Phi_0^*\eta + i\gamma'^*\Phi_1^*\Phi_+, \tag{A 1 b}$$

$$\eta_t + \nu'\eta + i\delta\eta = i\beta(\Phi_0\Phi_-^* + \Phi_0^*\Phi_+), \tag{A 1 c}$$

where Φ_1 denotes the amplitude of the $\cos(2m\pi x/l)$ acoustic mode and γ and γ' are complex interaction coefficients.

If we now examine the stability of the basic sound field with respect to a pair of side bands whose frequency gap is not related to the frequency of a water wave, we obtain equations of the forms (A 1 *a*) and (A 1 *b*), without the η term. The condition for stability of the basic sound field is that γ equals γ'; so, assuming the continuity of these interaction coefficients, we shall equate γ and γ' in equations (A 1).

Parallelling the analysis of § 4, we deduce that equations (A 1) have solutions with time dependence e^{ist} provided that s is a root of the characteristic equation

$$\{(s+\sigma-i\nu)^2 - \Delta^2 - |\gamma\Phi_1|^2\}\,(s+\delta-i\nu') + 2\alpha\beta\Delta\Phi_0\Phi_0^* + 2\alpha\beta\,\mathrm{Im}\,(\gamma\Phi_1\Phi_0^{*2}) = 0. \tag{A 2}$$

Once again it can be inferred that if the basic sound field is strong then, irrespective of the phases of α and β or the sign of Δ, there is at least one root s in the lower half of the complex plane. However, unlike the calculations we did in § 3, we need not exclude the case $\Delta = 0$. The extra terms in (A 2), as compared with (16), make it even more difficult to obtain the neutral-stability curves explicitly. However, on the lower branch of the neutral-stability curves, there is a significant range of Δ in which

$$|\gamma\Phi_1|^2 \ll \Delta^2,$$

and consequently the most important section of neutral-stability curve in the Φ_0, Δ plane is essentially independent of Φ_1. Of course, the relationship between Φ_0 and the amplitude of the driving mechanism is no longer of the form (15), but it can be determined either experimentally or by means of the theoretical procedures developed by Chester (1964).

REFERENCES

AMES, W. F., LEE, S. Y. & ZAISER, J. N. 1968 Nonlinear vibration of a travelling threadline. *Int. J. Nonlinear Mech.* **3**, 449–69.

CHESTER, W. 1964 Resonant oscillations in closed tubes. *J. Fluid Mech.* **18**, 44–64.

HUNTLEY, I. 1972 Observations on a spatial-resonance phenomenon. *J. Fluid Mech.* **53**, 209–216.

JEFFREYS, H. & JEFFREYS, B. S. 1962 *Methods of Mathematical Physics*. Cambridge University Press.

TADJBAKHSH, I. & KELLER, J. B. 1960 Standing surface waves of finite amplitude. *J. Fluid Mech.* **8**, 442–51.